A Checkerboard Problem and Modular Colorings of Graphs

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ABSTRACT

A modular k-coloring, $k \geq 2$, of a graph G without isolated vertices is a coloring of the vertices of G with the elements in $\mathbb{Z}_k$ (where adjacent vertices may be colored the same) having the property that for every two adjacent vertices of G, the sums of the colors of their neighbors are different in $\mathbb{Z}_k$. The minimum k for which G has a modular k-coloring is the modular chromatic number $\mathrm{mc}(G)$ of G. The modular chromatic number of a graph is at least as large as its chromatic number. The modular chromatic numbers of several well-known graphs are determined and a number of bounds are presented. For every nontrivial tree T, it is shown that $\mathrm{mc}(T) = 2$ or $\mathrm{mc}(T) = 3$. For every integer $r \geq 3$, it is shown that there exists an r-chromatic graph G with $\mathrm{mc}(G) = r + 1$. Several open problems are presented, including whether the modular chromatic number of every grid is 2, which has a checkerboard interpretation. Another open problem is whether there exists a planar graph with modular chromatic number 5.

Key Words: modular coloring, modular chromatic number.
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1 A Checkerboard Problem

The squares of an $m \times n$ checkerboard (m rows and n columns) are alternately colored black and red. Two squares are said to be *neighboring* if they belong to the same row or the same column and there is no square between them. Thus every two neighboring squares are of different colors. We are interested in the following conjecture.

The Checkerboard Conjecture *It is possible to place coins on some of the squares of an $m \times n$ checkerboard (at most one coin per square) such that for every two squares of the same color the numbers of coins on neighboring squares are of the same parity, while for every two squares of different colors the numbers of coins on neighboring squares are of opposite parity.*

Figure 1(a) shows a 7×8 checkerboard whose 56 squares are alternately colored black and red (where a shaded square represents a black square), while Figure 1(b) shows a placement of 20 coins on the checkerboard such that the number of coins on neighboring squares of every red square is even and the number of coins on neighboring squares of every black square is odd. Thus for every two squares of different colors, the numbers of coins on neighboring squares are of opposite parity. Consequently, the Checkerboard Conjecture is true for a 7×8 checkerboard.

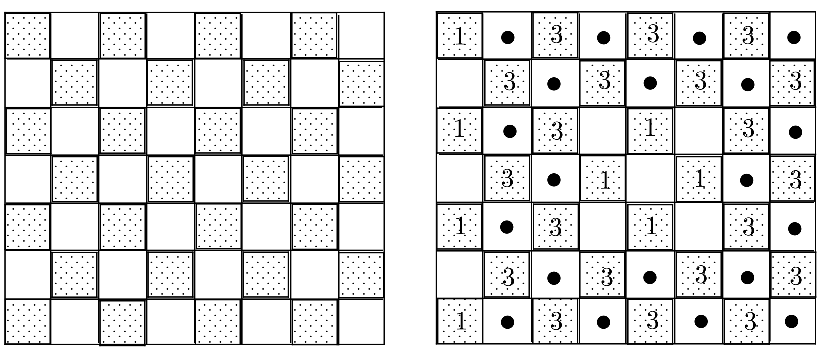

Figure 1: Coins on a 7×8 checkerboard

Observe that all 20 coins on the 7×8 checkerboard of Figure 1(b) are placed only on red squares. Thus the number of coins on neighboring squares of every red square is 0 and is therefore even. The number of coins on neighboring squares of each black square is odd (either 1 or 3) and this is shown in Figure 1(b) as well. Indeed, for any $m \times n$ checkerboard for which the Checkerboard Conjecture is true, there is always a solution in which all coins are placed only on squares of the same color.

We now indicate some pairs m, n of positive integers (with $m + n \geq 3$) for which the Checkerboard Conjecture is true.

Proposition 1.1 *The Checkerboard Conjecture is true if m and n are both odd (and $m \geq 3$ or $n \geq 3$).*

Proof. In each odd row k of the checkerboard (where $1 \leq k \leq m$), place a coin in column ℓ if (i) $k \equiv 1 \pmod 4$ and $\ell \equiv 1 \pmod 4$ or (ii) $k \equiv 3 \pmod 4$ and $\ell \equiv 3 \pmod 4$. See Figure 2 for a 5×7 checkerboard. ∎

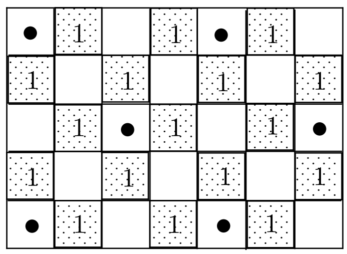

Figure 2: Coins on a 5×7 checkerboard

Proposition 1.2 *The Checkerboard Conjecture is true if $m = n$ (and $m = n \geq 2$).*

Proof. By Proposition 1.1, we need only be concerned with the situation where n is even. We consider two cases.

Case 1. $n \equiv 2 \pmod 4$. A coin is placed on a square in column ℓ of row k if

(i) $k < \ell$ and either (a) $k \equiv 1 \pmod 4$ and $\ell \equiv 2 \pmod 4$ or (b) $k \equiv 3 \pmod 4$ and $\ell \equiv 0 \pmod 4$, or

(ii) $k > \ell$ and either (c) $k \equiv 0 \pmod 4$ and $\ell \equiv 1 \pmod 4$ or (d) $k \equiv 2 \pmod 4$ and $\ell \equiv 3 \pmod 4$.

Case 2. $n \equiv 0 \pmod 4$. A coin is placed on a square in column ℓ of row k if

(i) $k < \ell$ and either (a) $k \equiv 1 \pmod 4$ and $\ell \equiv 0 \pmod 4$ or (b) $k \equiv 3 \pmod 4$ and $\ell \equiv 2 \pmod 4$, or

(ii) $k > \ell$ and either (c) $k \equiv 2 \pmod 4$ and $\ell \equiv 1 \pmod 4$ or (d) $k \equiv 0 \pmod 4$ and $\ell \equiv 3 \pmod 4$.

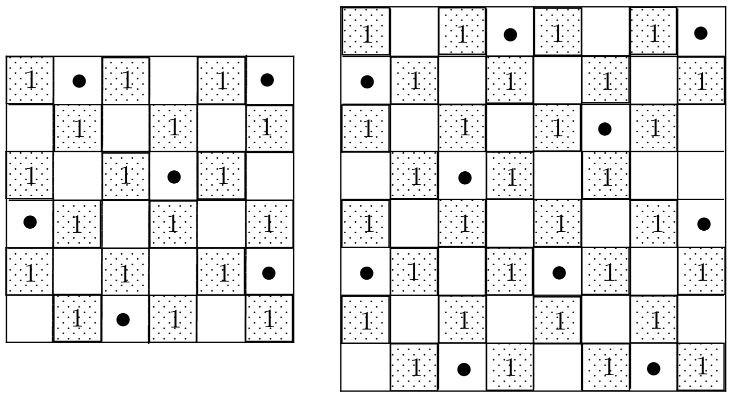

Figure 3: Coins on the 6×6 and 8×8 checkerboards

See Figure 3 for the 6×6 and 8×8 checkerboards. ■

We also state the following result without proof.

Proposition 1.3 *For every two integers m and n where $1 \leq m \leq 6$ and $m < n$, the Checkerboard Conjecture is true.*

The Checkerboard Problem described above can be stated in terms of graphs. For an $m \times n$ checkerboard, we can associate a graph G whose vertices are the squares of the checkerboard and where two vertices of G are adjacent if the corresponding squares are neighboring. Thus G is a bipartite graph of order mn, the partite sets of which are the set of black vertices and the set of red vertices. Determining whether the Checkerboard Conjecture is true for this checkerboard is equivalent to determining whether it is possible to color each vertex of G either 0 or 1 such that the sum of the colors (in $\mathbb{Z}_2$) of the neighboring vertices of each black vertex, say, is 1 and the sum of the colors (in $\mathbb{Z}_2$) of the neighboring vertices of each red vertex is 0. Of course, each black vertex can be colored 0 and so it is only a matter of determining whether there is an appropriate coloring of the red vertices of G with the colors 0 and 1.

We mentioned that the graph G that models the $m \times n$ checkerboard in the Checkerboard Problem is a bipartite graph of order mn. In fact, this graph is the Cartesian product $P_m \times P_n$ of paths of order m and n, which is commonly called a *grid*. The placement of coins on the 5×7 checkerboard shown in Figure 2 that provides a verification of the Checkerboard Conjecture in this case is then equivalent to the vertex coloring of the grid $P_5 \times P_7$ shown in Figure 4 using the element of $\mathbb{Z}_2$ as colors.

Since $G = P_m \times P_n$ is a connected bipartite graph, to show that the Checkerboard Conjecture is true for the $m \times n$ checkerboard, it suffices to

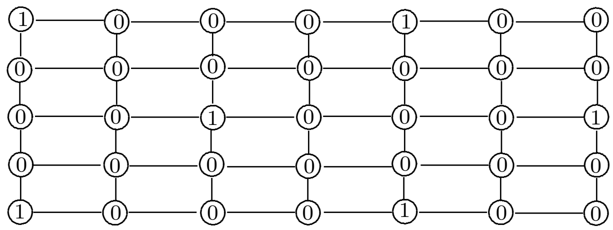

Figure 4: A coloring of $P_5 \times P_7$ with colors in $\mathbb{Z}_2$

show that for every two adjacent vertices of G, the sum in $\mathbb{Z}_2$ of the colors of their neighboring vertices are different. This observation suggests new vertex colorings for graphs in general.

2 Modular Colorings of Graphs

For a vertex v of a graph G, let $N(v)$ denote the neighborhood of v (the set of vertices adjacent to v). For a graph G without isolated vertices, let $c : V(G) \to \mathbb{Z}_k$ $(k \geq 2)$ be a vertex coloring of G where adjacent vertices may be colored the same. The *color sum* $\sigma(v)$ of a vertex v of G is defined as the sum of the colors of the vertices in $N(v)$, that is,

$$\sigma(v) = \sum_{u \in N(v)} c(u).$$

The coloring c is called a *modular sum k-coloring* or simply a *modular k-coloring* of G if $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_k$ for all pairs x, y of adjacent vertices of G. A coloring c is a *modular coloring* if c is a modular k-coloring for some integer $k \geq 2$. The *modular chromatic number* $\mathrm{mc}(G)$ of G is the minimum k for which G has a modular k-coloring. Thus for a given $m \times n$ checkerboard, the Checkerboard Conjecture states that $\mathrm{mc}(P_m \times P_n) = 2$. We refer to the book [1] for graph theory notation and terminology not described in this paper.

To illustrate the concepts introduced above, consider the bipartite graph G of Figure 5. Figure 5 also shows a modular 3-coloring of G (where the color of a vertex is placed within the vertex) together with the color sum $\sigma(v)$ for each vertex v of G (where the color sum of a vertex is placed next to the vertex). Thus $\mathrm{mc}(G) \leq 3$.

Next, we show that $\mathrm{mc}(G) \geq 3$. Assume, to the contrary, that there exists a modular 2-coloring c of G. By the symmetry of the graph G, we may assume that $\sigma(u_i) = 0$ and $\sigma(v_i) = 1$ for $1 \leq i \leq 4$. Since $\sigma(v_1) = 1$, it follows that $\{c(u_1), c(u_2)\} = \{0, 1\}$, which in turn implies that $c(u_3) = 0$

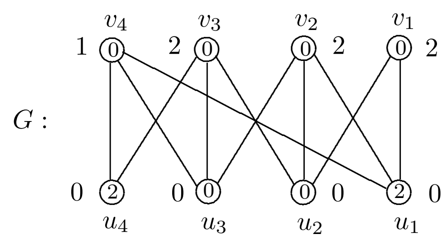

Figure 5: A bipartite graph G with $\mathrm{mc}(G) = 3$

since $\sigma(v_2) = 1$. Because $\sigma(v_3) = 1$, it follows that $c(u_4) \neq c(u_2)$ and so $c(u_4) = c(u_1)$. However then, $\sigma(v_4) = 0$, which is impossible. Therefore, $\mathrm{mc}(G) \geq 3$, which implies that $\mathrm{mc}(G) = 3$.

We first show that every nontrivial connected graph G has a modular coloring and so the modular chromatic number of G exists.

Proposition 2.1 *For every nontrivial connected graph G, there exists a modular k-coloring of G for some integer $k \geq 2$.*

Proof. Suppose that $V(G) = \{v_1, v_2, \ldots, v_n\}$. Define a coloring c of G by $c(v_i) = 2^{i-1}$ for $1 \leq i \leq n$. Let $k = \sum_{i=1}^{n-1} 2^i = 2(2^{n-1} - 1)$. Considering $c : V(G) \to \mathbb{Z}_k$, it follows that $1 \leq \sigma(v_i) \leq k$ for all i $(1 \leq i \leq n)$ and $\sigma(v_i) \neq \sigma(v_j)$ whenever v_i and v_j are adjacent. Hence c is a modular k-coloring of G. ∎

Corollary 2.2 *For every nontrivial connected graph G with $\Delta = \Delta(G)$,*

$$\mathrm{mc}(G) \leq 2^n - 2^{n-\Delta}.$$

Proof. For the coloring c in the proof of Proposition 2.1, the color sum of no vertex can exceed $\sum_{i=n-\Delta}^{n-1} 2^i = 2^n - 2^{n-\Delta}$, implying that $\mathrm{mc}(G) \leq 2^n - 2^{n-\Delta}$. ∎

The following observation permits us to restrict our attention to nontrivial connected graphs only.

Observation 2.3 *If G is a disconnected graph without isolated vertices consisting of components $G_1, G_2, \ldots, G_\ell$, then*

$$\mathrm{mc}(G) = \max\{\mathrm{mc}(G_i) : \ 1 \leq i \leq \ell\}.$$

The following two observations will also be useful to us.

Observation 2.4 *If u and v are two adjacent vertices in a graph G such that $N(u) - \{v\} = N(v) - \{u\}$, then $c(u) \neq c(v)$ for every modular coloring c of G.*

Observation 2.5 *If H is a complete subgraph of order k in a graph G, then* $\mathrm{mc}(G) \geq k$.

Obsevation 2.5 can be restated to say that $\mathrm{mc}(G) \geq \omega(G)$, where $\omega(G)$ denotes the clique number of G. In fact, there is an observation which is even stronger than that given in Observation 2.5.

Observation 2.6 *For every nontrivial connected graph G,* $\mathrm{mc}(G) \geq \chi(G)$.

As the graph G of Figure 5 shows, the inequality in Observation 2.6 can be strict.

3 On Graphs with Small Modular Chromatic Numbers

We first consider cycles. Figure 6 shows a modular 2-coloring of C_8 and modular 3-colorings of C_9, C_{10}, and C_{11}.

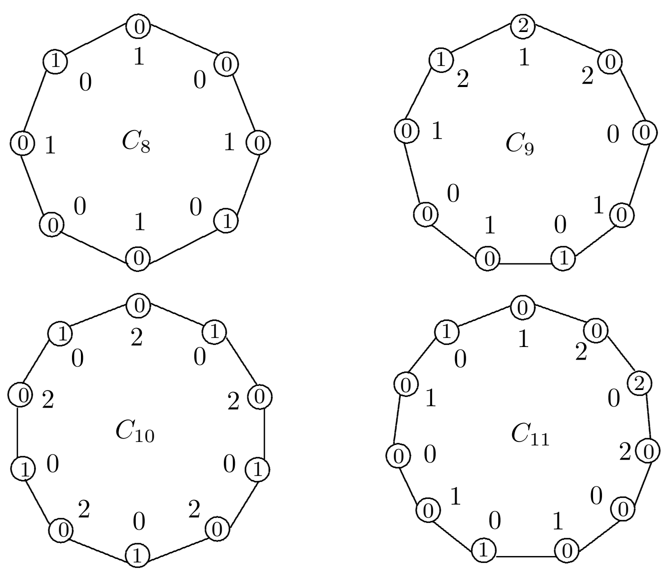

Figure 6: Modular colorings of C_n for $8 \leq n \leq 11$

It turns out that the modular colorings of the four cycles in Figure 6 illustrate the modular chromatic numbers for all cycles.

Proposition 3.1 *For each integer $n \geq 3$,*

$$\mathrm{mc}(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 4 \\ 3 & \text{otherwise.} \end{cases}$$

Proof. Suppose that $C_n : v_1, v_2, \ldots, v_n, v_1$. We consider three cases.

Case 1. $n \equiv 0 \pmod 4$. Since n is even, $\mathrm{mc}(C_n) \geq 2$. Because the coloring $c : V(C_n) \to \mathbb{Z}_2$ defined by

$$c(v_i) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod 4 \\ 0 & \text{if } i \not\equiv 0 \pmod 4 \end{cases}$$

is a modular 2-coloring, it follows that $\mathrm{mc}(C_n) \leq 2$ and so $\mathrm{mc}(C_n) = 2$.

Case 2. $n \equiv 2 \pmod 4$. Again, $\mathrm{mc}(C_n) \geq 2$. Because the coloring $c : V(C_n) \to \mathbb{Z}_3$ defined by

$$c(v_i) = \begin{cases} 0 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even} \end{cases}$$

is a modular 3-coloring, it follows that $\mathrm{mc}(C_n) \leq 3$. We show that $\mathrm{mc}(C_n) \neq 2$. Assume, to the contrary, that there exists a modular 2-coloring $c' : V(C_n) \to \mathbb{Z}_2$ of C_n. Then we may assume that

$$\sigma(v_i) = \begin{cases} 0 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even.} \end{cases}$$

We may further assume that $c'(v_1) = 1$ and $c'(v_3) = 0$. Since $\sigma(v_4) = 1$, it follows that $c'(v_5) = 1$. Continuing in this manner, we see that $c'(v_i) = 1$ for all i with $1 \leq i \leq n$ for which $i \equiv 1 \pmod 4$. However then, $\sigma(v_n) = 0$, which is a contradiction. Thus $\mathrm{mc}(C_n) = 3$.

Case 3. *n is odd.* Thus $\mathrm{mc}(C_n) \geq 3$. Since the coloring $c : V(C_n) \to \mathbb{Z}_3$ defined by

$$c(v_i) = \begin{cases} 0 & \text{if } i \not\equiv 0 \pmod 4 \text{ and } i < n \\ 1 & \text{if } i \equiv 0 \pmod 4 \\ 2 & \text{if } i = n \end{cases}$$

is a modular 3-coloring, it follows that $\mathrm{mc}(C_n) \leq 3$ and so $\mathrm{mc}(C_n) = 3$. ■

While each nontrivial path is a tree with modular chromatic number 2, not every tree has modular chromatic number 2. We show for the tree T in Figure 7(a) that $\mathrm{mc}(T) = 3$. Assume, to the contrary, that $\mathrm{mc}(T) = 2$. Then there exists a modular 2-coloring c of T. Because of the symmetry of the structure of T, the color sums of the vertices of T are those shown in Figure 7(b). Since $\sigma(w_2) = \sigma(v_8) = 1$, it follows that $c(v_5) = c(v_7) = 1$. This, however, contradicts the fact that $\sigma(v_6) = 1$. Hence $\mathrm{mc}(T) \neq 2$ and so $\mathrm{mc}(T) = 3$. A modular 3-coloring of T is shown in Figure 7(c).

On the other hand, every nontrivial tree has modular chromatic number 2 or 3.

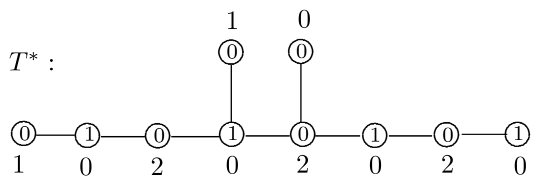

Figure 7: A tree T with $\mathrm{mc}(T) = 3$

Theorem 3.2 *If T is a nontrivial tree, then* $\mathrm{mc}(T) = 2$ *or* $\mathrm{mc}(T) = 3$.

Proof. Since $\mathrm{mc}(T) \geq \chi(T) = 2$ for every nontrivial tree T, it remains to show that there is a modular 3-coloring of T. Since $\mathrm{mc}(T) = 2$ if T is a nontrivial star, suppose that $\mathrm{diam}(T) \geq 3$. Let $x \in V(T)$ be an end-vertex with $N(x) = \{y\}$. For each i with $0 \leq i \leq e(x)$, where $e(x)$ is the eccentricity of x, let $V_i = \{v \in V(T) : d(v, x) = i\}$. Thus $V_0 = \{x\}$, $V_1 = \{y\}$, and $\{V_0, V_1, \ldots, V_{e(x)}\}$ is a partition of $V(T)$. We now define a modular 3-coloring c of T. First define $c(v) = 0$ if $v \in V_i$ where $0 \leq i \leq e(x)$ and i is even. Also let $c(y) = 1$. Then $\sigma(x) = 1$ and $\sigma(u) = 0$ for each $u \in V_j$ where $1 \leq j \leq e(x)$ and j is odd.

Since T is not a star, $e(x) \geq 3$. Suppose that $c(v)$ has been defined for all vertices $v \in V_i$, where $0 \leq i \leq 2k < e(x)$ for some positive integer k, such that $c(v) = 0$ if and only if $v \in V_i$ and i is even. If $w \in V_{2k}$ is not an end-vertex, then let $N(w) \cap V_{2k-1} = \{w_0\}$ and $N(w) \cap V_{2k+1} = \{w_1, w_2, \ldots, w_{d-1}\}$, where $d = \deg w \geq 2$. Then define $c(w_i) = 1$ for $1 \leq i \leq d-1$ if $\sum_{i=0}^{d-1} c(w_i) \not\equiv 0 \pmod 3$. Otherwise, let $c(w_1) = 2$ and $c(w_i) = 1$ for $2 \leq i \leq d-1$ (if $d \geq 3$). This results in a modular 3-coloring of T and so $\mathrm{mc}(T) \leq 3$. ■

Of course, every nontrivial tree is a bipartite graph. Furthermore, the modular chromatic number of every nontrivial bipartite graph is at least 2. We saw in Figure 5 that a bipartite graph that is not a tree may also have modular chromatic number 3. Whether a bipartite graph G can have a larger modular chromatic number is not known, but it can never exceed $\Delta(G)$ by more than 1. If G is a bipartite graph with $\Delta(G) = \Delta$, then we assign 0 to each vertex in one partite set of G and 1 to each vertex in the other partite set of G, producing a modular coloring using the colors on $\mathbb{Z}_{1+\Delta}$. Therefore, we have the following observation.

Observation 3.3 *If G is a bipartite graph, then* $\mathrm{mc}(G) \leq 1 + \Delta(G)$.

There are some conditions that are sufficient for a bipartite graph to have modular chromatic number 2.

Lemma 3.4 *Let G be a bipartite graph. If G contains a vertex that is adjacent to all vertices in a partite set, then* $\mathrm{mc}(G) = 2$.

Proof. Suppose that the partite sets of G are V_1 and V_2 and $v \in V_1$ is adjacent to every vertex in V_2. Then the coloring $c : V(G) \to \mathbb{Z}_2$ defined by $c(v) = 1$ and $c(x) = 0$ for all $x \in V(G) - \{v\}$ is a modular 2-coloring and so $\mathrm{mc}(G) = 2$. ■

Lemma 3.5 *If G is a bipartite graph such that one of its partite sets consists only of odd vertices, then* $\mathrm{mc}(G) = 2$.

Proof. Suppose that the partite sets of G are V_1 and V_2 and every vertex in V_1 has odd degree. Then the coloring $c : V(G) \to \mathbb{Z}_2$ defined by $c(v) = 0$ if and only if $v \in V_1$ is a modular 2-coloring and so $\mathrm{mc}(G) = 2$. ■

Proposition 3.6 *If G is a bipartite graph the degrees of whose vertices are of the same parity, then* $\mathrm{mc}(G \times K_2) = 2$.

Proof. Since G is a bipartite graph, $G \times K_2$ is also bipartite. If all vertices of G are even, then all vertices of $G \times K_2$ are odd and so $\mathrm{mc}(G \times K_2) = 2$ by Lemma 3.5. Thus we may assume that all vertices of G are odd. In this case, each vertex of $G \times K_2$ is even. Suppose that G is of order n and let G_1 and G_2 be two copies of G in $G \times K_2$, where $V(G_1) = \{u_1, u_2, \ldots, u_n\}$, $V(G_2) = \{w_1, w_2, \ldots, w_n\}$, and $u_i w_i \in E(G)$ for $1 \le i \le n$. Consider the coloring $c : V(G) \to \mathbb{Z}_2$ such that (i) $c(u) = 0$ for every $u \in V(G_1)$ and (ii) $c : V(G_2) \to \mathbb{Z}_2$ is a proper coloring of G_2. Since $\sigma(u_i) = c(w_i)$ and $\sigma(w_i) = c(w_i) + 1$ for $1 \le i \le n$, it follows that $\sigma(u_i) \ne \sigma(w_i)$. Furthermore, if u_i and u_j are adjacent, then w_i and w_j are also adjacent and so $c(w_i) \ne c(w_j)$, implying that $\sigma(u_i) \ne \sigma(u_j)$. Similarly, if w_i and w_j are adjacent, then $\sigma(w_i) \ne \sigma(w_j)$. Therefore, c is a modular 2-coloring of $G \times K_2$. ■

A well-known class of bipartite graphs are the n-cubes. By Proposition 3.6, all of these graphs have modular chromatic number 2.

Corollary 3.7 *For every positive integer n,* $\mathrm{mc}(Q_n) = 2$.

We saw in Proposition 3.1 that $\mathrm{mc}(C_n) = 3$ if $n \ge 6$ and $n \equiv 2 \pmod 4$. Hence the bound given in Observation 3.3 is sharp when $\Delta(G) = 2$. Whether this bound is sharp when $\Delta(G) \ge 3$ is not known, as was noted earlier. The upper bound for $\mathrm{mc}(G)$ stated in Observation 3.3 can be extended to graphs whose chromatic number is at least 3.

Theorem 3.8 *If G is a k-chromatic graph ($k \ge 2$) with maximum degree Δ, then*

$$\mathrm{mc}(G) \le \Delta(\Delta + 1)^{k-2} + 1.$$

Proof. Let $c : V(G) \to \mathbb{N}_k$ be a proper k-coloring of G. We define another coloring c' of G by

$$c'(v) = \begin{cases} 0 & \text{if } c(v) = 1 \\ (\Delta+1)^{c(v)-2} & \text{if } 2 \le c(v) \le k. \end{cases}$$

We show that c' is a modular $\big(\Delta(\Delta+1)^{k-2}+1\big)$-coloring of G.

Let $V_1, V_2, \ldots, V_k$ be the color classes of G resulting from the coloring c. If x and y are adjacent vertices in G, then $x \in V_s$ and $y \in V_t$ for some s and t with $s \neq t$. Also, let $a_i = |N(x) \cap V_i|$ and $b_i = |N(y) \cap V_i|$ for $1 \le i \le k$. Then $\sum_{i=1}^{k} a_i = \deg x$ and $\sum_{i=1}^{k} b_i = \deg y$ and furthermore, $a_s = b_t = 0$ and $a_t, b_s \ge 1$.

Let $p = \max\{i \in \mathbb{N}_k : a_i \neq b_i\}$ and $\alpha = \sum_{i=p+1}^{k} a_i(\Delta+1)^{i-2}$ if $p < k$ while $\alpha = 0$ if $p = k$. In addition, we may assume that $a_p < b_p$. Since $s, t \le p$, it follows that $p \ge 2$. If $p = 2$, then $a_2 = b_1 = 0$ and so $x \in V_2$ and $y \in V_1$. Then $\sigma(x) = \alpha < \alpha + c'(x) \le \sigma(y)$. If $3 \le p \le k$, then

$$\begin{aligned}
\sigma(x) &= \sum_{i=2}^{k} a_i(\Delta+1)^{i-2} = \sum_{i=2}^{p-1} a_i(\Delta+1)^{i-2} + a_p(\Delta+1)^{p-2} + \alpha \\
&\le \sum_{i=1}^{p-1} a_i(\Delta+1)^{p-3} + a_p(\Delta+1)^{p-2} + \alpha \le \Delta(\Delta+1)^{p-3} + a_p(\Delta+1)^{p-2} + \alpha \\
&< (\Delta+1)^{p-2} + a_p(\Delta+1)^{p-2} + \alpha = (a_p+1)(\Delta+1)^{p-2} + \alpha,
\end{aligned}$$

while

$$\sigma(y) = \sum_{i=1}^{k} b_i(\Delta+1)^{i-2} \ge \sum_{i=p}^{k} b_i(\Delta+1)^{i-2} \ge (a_p+1)(\Delta+1)^{p-2} + \alpha,$$

that is, $\sigma(x) < \sigma(y)$. Therefore, c' is a modular $\big(\Delta(\Delta+1)^{k-2}+1\big)$-coloring of G and so $\mathrm{mc}(G) \le \Delta(\Delta+1)^{k-2}+1$. ■

4 On Graphs with Large Modular Chromatic Numbers

Certainly every complete graph of order n has modular chromatic number n. This observation can be extended to a larger class of graphs.

Proposition 4.1 *If G is a complete multipartite graph, then* $\mathrm{mc}(G) = \chi(G)$.

Proof. Suppose that G is a complete k-partite graph, where $k \geq 2$. Then $\text{mc}(G) \geq k$. Let $V_1, V_2, \ldots, V_k$ be the partite sets of G and let $v_i \in V_i$ for $1 \leq i \leq k$. Define the coloring $c : V(G) \to \mathbb{Z}_k$ by $c(v_i) = i - 1$ $(1 \leq i \leq k)$ and $c(v) = 0$ for $v \in V(G) - \{v_1, v_2, \ldots, v_k\}$. Then for each $v_j \in V(G)$ $(1 \leq j \leq k)$, it follows that $\sigma(v_j) = \left(\sum_{i=1}^{k-1} i\right) - (j-1) = \binom{k}{2} - (j-1)$. Therefore, c is a modular k-coloring and so $\text{mc}(G) = k$. ■

We now consider a class of graphs G where $\chi(G)$ is large and for which $\text{mc}(G)$ may exceed $\chi(G)$. Since $\chi(K_r \times K_2) = r$ for each integer $r \geq 2$, it follows that $\text{mc}(K_r \times K_2) \geq r$. It is easy to see that $\text{mc}(K_1 \times K_2) = 2$ and $\text{mc}(K_2 \times K_2) = 2$. We show that the 3-chromatic graph $K_3 \times K_2$ has modular chromatic number 4. Construct $G = K_3 \times K_2$ from two disjoint copies of K_3 with vertex sets $\{u_1, u_2, u_3\}$ and $\{w_1, w_2, w_3\}$ by joining u_i and w_i for $1 \leq i \leq 3$. Observe that the coloring $c^* : V(G) \to \mathbb{Z}_4$ defined by $c^*(u_i) = 0$ and $c^*(w_i) = i - 1$ for $1 \leq i \leq 3$ is a modular 4-coloring. Assume, to the contrary, that there exists a modular 3-coloring c of G. We may assume that $\sigma(u_1) = \sigma(w_2)$, $\sigma(u_2) = \sigma(w_3)$, and $\sigma(u_3) = \sigma(w_1)$. Since $\sigma(u_1) = \sigma(w_2)$, it follows that $c(u_2) + c(u_3) + c(w_1) = c(u_2) + c(w_1) + c(w_3)$ and so $c(u_3) = c(w_3)$. Similarly, $c(u_1) = c(w_1)$ and $c(u_2) = c(w_2)$. However then, $\sigma(u_1) = c(u_2) + c(u_3) + c(w_1) = c(w_2) + c(w_3) + c(u_1) = \sigma(w_1)$, which is impossible. Therefore, $\text{mc}(K_3 \times K_2) = 4$ as claimed. Moreover, $\text{mc}(K_6 \times K_2) = 6$, as the modular 6-coloring of $K_6 \times K_2$ in Figure 8 shows.

$K_6 \times K_2$:

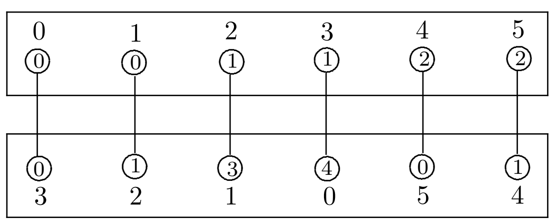

Figure 8: A modular 6-coloring of $K_6 \times K_2$

Therefore, $\text{mc}(K_3 \times K_2) = \chi(K_3 \times K_2) + 1$ while $\text{mc}(K_6 \times K_2) = \chi(K_6 \times K_2)$. In fact, $\text{mc}(K_r \times K_2) \in \{r, r+1\}$ for every positive integer r and $\text{mc}(K_r \times K_2) = r$ only when $r \equiv 2 \pmod 4$. We verify this next.

Theorem 4.2 *For every positive integer r,*

$$r \leq \text{mc}(K_r \times K_2) \leq r + 1.$$

Furthermore, $\text{mc}(K_r \times K_2) = r$ *if and only if* $r \equiv 2 \pmod 4$.

Proof. First, we show that if $r \equiv 2 \pmod 4$, then $\text{mc}(K_r \times K_2) = r$. Construct $G = K_r \times K_2$ with $V(G) = U \cup W$, where $U = \{u_1, u_2, \ldots, u_r\}$

and $W = \{w_1, w_2, \dots, w_r\}$ are disjoint sets and $\langle U \rangle \cong \langle W \rangle \cong K_r$, by joining u_i to w_i for $1 \le i \le r$. Write $r = 4\ell + 2$, where ℓ is a nonnegative integer. It suffices to verify that $\mathrm{mc}(G) \le 4\ell + 2$.

Consider the coloring $c : V(G) \to \mathbb{Z}_{4\ell+2}$ defined by $c(u_i) = i - 1$ and $c(w_i) = 0$ for $1 \le i \le 4\ell + 2$. Observe that

$$\sum_{i=1}^{4\ell+2} c(u_i) = 0 + 1 + \cdots + (4\ell+1) = (4\ell+1)(2\ell+1) = (-1)(2\ell+1) = 2\ell+1.$$

Therefore,

$$\begin{aligned} \sigma(u_i) &= (2\ell+1) - c(u_i) + c(w_i) = 2\ell + 2 - i \\ \sigma(w_i) &= c(u_i) = i - 1. \end{aligned}$$

This implies that $\sigma(u_i) = \sigma(u_j)$ if and only if $i = j$ and similarly, $\sigma(w_i) = \sigma(w_j)$ if and only if $i = j$. Also, since $\{2i : 1 \le i \le 4\ell + 2\} = \{0, 2, \dots, 4\ell\}$, it follows that $2\ell + 3 \ne 2i$ and so

$$\sigma(u_i) = 2\ell + 2 - i \ne i - 1 = \sigma(w_i)$$

for $1 \le i \le 4\ell + 2$. Therefore, c is a modular $(4\ell + 2)$-coloring of G.

Next, we show that if $r \not\equiv 2 \pmod 4$, then $\mathrm{mc}(K_r \times K_2) = r + 1$. Since the result is trivial for $r = 1$, assume that $r \ge 3$. Construct $G = K_r \times K_2$ with $V(G) = U \cup W$, where $U = \{u_1, u_2, \dots, u_r\}$ and $W = \{w_1, w_2, \dots, w_r\}$ are disjoint sets and $\langle U \rangle \cong \langle W \rangle \cong K_r$, by joining u_i to w_i for $1 \le i \le r$. We first show that $\mathrm{mc}(G) \le r + 1$ by finding a modular $(r + 1)$-coloring of G. We consider two cases.

Case 1. $r \equiv 0 \pmod 4$. Then we may write $r = 4\ell$, where ℓ is a positive integer. Consider the coloring $c : V(G) \to \mathbb{Z}_{4\ell+1}$ defined by $c(u_i) = i - 1$ and $c(w_i) = 4\ell$ for $1 \le i \le 4\ell$. Observe that

$$\sum_{i=1}^{4\ell} c(u_i) = 0 + 1 + \cdots + (4\ell - 1) = (4\ell - 1)(2\ell) = (-2)(2\ell) = 1.$$

Therefore,

$$\begin{aligned} \sigma(u_i) &= 1 - c(u_i) + c(w_i) = 1 - (i - 1) + 4\ell = 1 - i \\ \sigma(w_i) &= (4\ell - 1)(4\ell) + c(u_i) = (-2)(-1) + (i - 1) = 1 + i. \end{aligned}$$

This implies that $\sigma(u_i) = \sigma(u_j)$ if and only if $i = j$ and similarly, $\sigma(w_i) = \sigma(w_j)$ if and only if $i = j$. Furthermore, observe that $\{2i : 1 \le i \le 4\ell\} = \mathbb{Z}_{4\ell+1} - \{0\}$ and so $2i \ne 0$, implying that

$$\sigma(u_i) = 1 - i \ne 1 + i = \sigma(w_i)$$

for $1 \leq i \leq 4\ell$. This verifies that c is a modular $(4\ell+1)$-coloring of $K_{4\ell} \times K_2$.

Case 2. $r \equiv 1, 3 \pmod 4$. Then r is odd and so $r = 2\ell + 1$ for some positive integer ℓ. Consider the coloring $c : V(G) \to \mathbb{Z}_{2\ell+2}$ defined by $c(u_i) = i - 1$ and $c(w_i) = \ell + 1$ for $1 \leq i \leq 2\ell + 1$. Observe that

$$\sum_{i=1}^{2\ell+1} c(u_i) = 0 + 1 + \cdots + 2\ell = (2\ell + 1)\ell = (-1)\ell = \ell + 2$$

Therefore,

$$\sigma(u_i) = (\ell + 2) - c(u_i) + c(w_i) = (\ell + 2) - (i - 1) + (\ell + 1) = 2 - i$$
$$\sigma(w_i) = (2\ell)(\ell + 1) + c(u_i) = \ell(2\ell + 2) + (i - 1) = i - 1.$$

This implies that $\sigma(u_i) = \sigma(u_j)$ if and only if $i = j$ and similarly, $\sigma(w_i) = \sigma(w_j)$ if and only if $i = j$. Furthermore, observe that $\{2i : 1 \leq i \leq 2\ell+1\} = \{0, 2, \ldots, 2\ell\}$, that is, $2i \neq 3$. Hence

$$\sigma(u_i) = 2 - i \neq i - 1 = \sigma(w_i)$$

for $1 \leq i \leq 2\ell + 1$. This verifies that c is a modular $(2\ell + 2)$-coloring of $K_{2\ell+1} \times K_2$.

We next show that there is no modular r-coloring of $K_r \times K_2$ if $r \not\equiv 2 \pmod 4$. Assume, to the contrary, that $c' : V(G) \to \mathbb{Z}_r$ is a modular r-coloring of G. Also, let $\alpha = \sum_{i=1}^{r} c'(u_i)$ and $\beta = \sum_{i=1}^{r} c'(w_i)$. Since $\sigma(u_i) = \alpha - c'(u_i) + c'(w_i)$ for $1 \leq i \leq r$, it follows that

$$\sum_{i=1}^{r} \sigma(u_i) = (r - 1)\alpha + \beta = -\alpha + \beta = -2\alpha + (\alpha + \beta). \tag{1}$$

Also,

$$\sigma(u_i) + \sigma(w_i) = [\alpha - c'(u_i) + c'(w_i)] + [\beta - c'(w_i) + c'(u_i)] = \alpha + \beta \tag{2}$$

for $1 \leq i \leq r$. We again consider two cases.

Case 1. $r \equiv 0 \pmod 4$. Then $r = 4\ell$ for some positive integer ℓ. Since c' is a modular coloring, $\{\sigma(u_i) : 1 \leq i \leq 4\ell\} = \mathbb{Z}_{4\ell}$ and so

$$\sum_{i=1}^{4\ell} \sigma(u_i) = 0 + 1 + \cdots + (4\ell - 1) = (4\ell - 1)(2\ell) = (-1)(2\ell) = 2\ell. \tag{3}$$

By (1) and (3),

$$\alpha + \beta = 2\ell + 2\alpha = 2(\ell + \alpha). \tag{4}$$

Since $\{\sigma(u_i) : 1 \le i \le 4\ell\} = \mathbb{Z}_{4\ell}$, there exists a vertex u_{i^*} in U such that $\sigma(u_{i^*}) = \ell + \alpha$ and so by (2) and (4),

$$(\ell + \alpha) + \sigma(w_{i^*}) = \sigma(u_{i^*}) + \sigma(w_{i^*}) = \alpha + \beta = 2(\ell + \alpha).$$

However then, $\sigma(w_{i^*}) = \ell + \alpha$ as well, which is impossible. Hence, there is no such c'.

Case 2. $r \equiv 1, 3 \pmod 4$. Then $r = 2\ell + 1$ for some positive integer ℓ. Since c' is a modular coloring, $\{\sigma(u_i) : 1 \le i \le 2\ell + 1\} = \mathbb{Z}_{2\ell+1}$ and so

$$\sum_{i=1}^{2\ell+1} \sigma(u_i) = 0 + 1 + \cdots + 2\ell = \ell(2\ell + 1) = 0. \qquad (5)$$

Hence, $\alpha + \beta = 2\alpha$ by (1) and (5). Since $\{\sigma(u_i) : 1 \le i \le 2\ell + 1\} = \mathbb{Z}_{2\ell+1}$, there exists a vertex u_{i^*} in U such that $\sigma(u_{i^*}) = \alpha$. Then by an argument similar to the one used in Case 1, it follows that $\sigma(w_{i^*}) = \alpha = \sigma(u_{i^*})$, which is again a contradiction. Hence, there is no such c'.

Thus, $\mathrm{mc}(K_r \times K_2) \ge r + 1$, completing the proof. ■

In Figure 5 we saw a bipartite graph G of order 8 with $\mathrm{mc}(G) = 3$. This graph is clearly a subgraph of the complete bipartite graph $K_{4,4}$ which has modular chromatic number 2 by Proposition 4.1. This example illustrates an interesting feature for the modular chromatic number of a graph, namely, if H is a subgraph of a graph G, then it is possible that $\mathrm{mc}(H) > \mathrm{mc}(G)$. In fact, even if H is an induced subgraph of a graph G, then $\mathrm{mc}(H) > \mathrm{mc}(G)$ is possible. For example, $\mathrm{mc}(K_3 \times K_2) = 4$ by Theorem 4.2. However, $K_3 \times K_2$ is an induced subgraph of the graph of Figure 9 which has modular chromatic number 3.

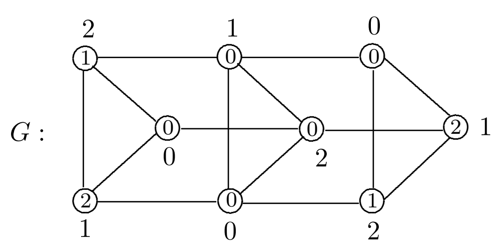

Figure 9: A modular 3-coloring of a graph G

5 Modular Colorings of Planar Graphs

We have already stated the modular chromatic numbers of some familiar planar graphs. For example, for every nontrivial tree T, either $\mathrm{mc}(T) = 2$ or mc$(T) = 3$. In the case of cycles, we also have mc$(C_n) = 2$ or mc$(C_n) = 3$, where mc$(C_n) = 2$ if and only if if $n \equiv 0 \pmod 4$. We now consider some classes of planar graphs that are the joins of some common graphs.

For $n \geq 3$, the wheel W_n is the graph $C_n + K_1$.

Proposition 5.1 *For each integer $n \geq 3$, mc$(W_n) = \chi(W_n)$.*

Proof. Construct $W_n = C_n + K_1$ from an n-cycle $C_n : u_1, u_2, \ldots, u_n, u_1$ by adding a new vertex w and joining w to every vertex of C_n. If $n = 3$, then $W_4 = K_4$ and the result immediately follows from Proposition 4.1. Thus, suppose that $n \geq 4$. We first show that mc$(W_n) = 3$ if n is even. Consider the coloring $c : V(W_n) \to \mathbb{Z}_3$ defined by

$$c(v) = \begin{cases} 0 & \text{if } v = w \\ 1 & \text{if } v = u_i, \text{ where } i \text{ is odd} \\ 2 & \text{if } v = u_i, \text{ where } i \text{ is even.} \end{cases}$$

Observe that c is a proper coloring and furthermore, $\sigma(v) = c(v)$ for every v in W_n. Therefore, c is a modular 3-coloring of W_n.

Next we show that mc$(W_n) = 4$ if n is odd. We consider the following two cases.

Case 1. $n \equiv 1 \pmod 4$. Then $n = 4k + 1$ for some positive integer k. Define a coloring $c : V(W_n) \to \mathbb{Z}_4$ by

$$c(v) = \begin{cases} k+3 & \text{if } v = w \\ 2 & \text{if } v = u_{4k+1} \\ 1 & \text{if } v = u_i, \text{ where } i \in \{4, 8, \ldots, 4k\} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sigma(v) = \begin{cases} k+2 & \text{if } v = w \\ k+1 & \text{if } v \in \{u_1, u_{4k}\} \\ k+3 & \text{if } v = u_i, \text{ where } i \in \{2, 4, \ldots, 4k-2\} \\ k & \text{if } v = u_i, \text{ where } i \in \{3, 5, \ldots, 4k+1\} \end{cases}$$

and so c is a modular 4-coloring of W_n.

Case 2. $n \equiv 3 \pmod 4$. Then $n = 4k + 3$ for some positive integer k. Define a coloring $c : V(W_n) \to \mathbb{Z}_4$ by

$$c(v) = \begin{cases} k+3 & \text{if } v = w \\ 2 & \text{if } v = u_{4k+3} \\ 1 & \text{if } v = u_i, \text{ where } i \in \{4, 8, \ldots, 4k\} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sigma(v) = \begin{cases} k+2 & \text{if } v = w \\ k+1 & \text{if } v \in \{u_1, u_{4k+2}\} \\ k+3 & \text{if } v = u_i, \text{ where } i \in \{2,4,\ldots,4k\} \cup \{4k+3\} \\ k & \text{if } v = u_i, \text{ where } i \in \{3,5,\ldots,4k+1\}. \end{cases}$$

Therefore, c is a modular 4-coloring of W_n. ■

As a consequence of Proposition 5.1, we obtain an infinite class of maximal planar graphs G with $\mathrm{mc}(G) = \chi(G)$.

Proposition 5.2 *For each integer $n \geq 3$, $\mathrm{mc}(C_n + \overline{K}_2) = \chi(C_n + \overline{K}_2) = \chi(W_n)$.*

Proof. For a given integer $n \geq 3$, suppose that $\mathrm{mc}(W_n) = k$. Using the same notation in the proof of Proposition 5.1, we assume that W_n is constructed by joining a vertex w to each vertex of the cycle C_n : $u_1, u_2, \ldots, u_n, u_1$. Thus by Proposition 5.1, $\mathrm{mc}(W_n) = 3$ if n is even and $\mathrm{mc}(W_n) = 4$ if n is odd. To construct $G = C_n + \overline{K}_2$, we add a new vertex x to W_n and join x to each vertex of C_n. Let $c : V(W_n) \to \mathbb{Z}_k$ be the modular k-coloring of W_n given in the proof of Proposition 5.1. Now extend the coloring c to G by defining $c(x) = 0$. Then $\sigma(x) = \sigma(w)$. Since $\sigma(w) \neq \sigma(u_i)$ for each integer i with $1 \leq i \leq n$, it follows that $\sigma(x) \neq \sigma(u_i)$ as well. Further, since $\sigma_G(u_i) = \sigma_{W_n}(u_i)$ for $1 \leq i \leq n$, it follows that c is a modular k-coloring of G, giving the desired result. ■

Another familiar class of maximal planar graphs are the graphs $P_n + K_2$ for $n \geq 2$.

Proposition 5.3 *For each integer $n \geq 2$, $\mathrm{mc}(P_n + K_2) = 4$.*

Proof. Let $G = P_n + K_2$, where $V(K_2) = \{u, w\}$ and $P_n : v_1, v_2, \ldots, v_n$. Since $\chi(P_n + K_2) = 4$, it follows that $\mathrm{mc}(P_n + K_2) \geq 4$.

Next, we show that $\mathrm{mc}(P_n + K_2) \leq 4$. Define a modular 2-coloring $c : V(P_n) \to \mathbb{Z}_2$ of P_n as follows. If $n \not\equiv 0 \pmod 4$, then

$$c(v_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod 4 \\ 0 & \text{if } i \not\equiv 1 \pmod 4. \end{cases}$$

If $n \equiv 0 \pmod 4$, then

$$c(v_i) = \begin{cases} 1 & \text{if } i \equiv 2 \pmod 4 \\ 0 & \text{if } i \not\equiv 2 \pmod 4. \end{cases}$$

Define a coloring $c' : V(G) \to \mathbb{Z}_4$ by $c'(u) = 1$, $c'(w) = 3$, and $c'(v_i) = 2c(v_i)$. (See Figure 10 for modular 4-colorings of $P_4 + K_2$ and $P_5 + K_2$.) Since c' is a modular 4-coloring, $\mathrm{mc}(G) = 4$. ■

We have now seen that $\mathrm{mc}(G) \leq 4$ for every planar graph that we have encountered in this paper. This suggests the following problem.

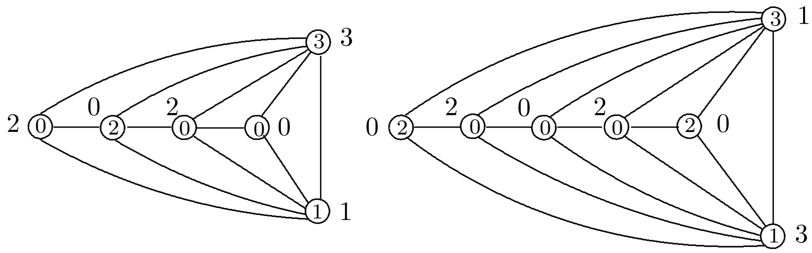

Figure 10: Modular 4-colorings of $P_4 + K_2$ and $P_5 + K_2$

Problem 5.4 *Does there exist a planar graph whose modular chromatic number is* 5*?*

If the answer to this question is no (and we can verify this), then there is a new Four Color Theorem for which the classic Four Color Theorem is a corollary.

We close with three additional problems.

Problem 5.5 *Is there a constant* C *such that* $\text{mc}(G) \leq C$ *for every bipartite graph* G*?*

Problem 5.6 *Is there a graph* G *such that* $\omega(G) < \chi(G) < \text{mc}(G)$*?*

Problem 5.7 *Is there a graph* G *such that* $\text{mc}(G) \geq \chi(G) + 2$*? Is there an upper bound for* $\text{mc}(G)$ *in terms of* $\chi(G)$*?*

References

[1] G. Chartrand and L. Lesniak, *Graphs & Digraphs: Fourth Edition.* Chapman & Hall/CRC, Boca Raton, FL (2005).